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PROGRAMMED INSTRUCTION - PROJECT 1000

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INTRODUCTION OF PROGRAMMED INSTRUCTION

Ideas, methods and techniques which are new or different in education are apt to create controversy and to be resisted. Programmed instruction is no exception.

Debate and argument will not solve the questions of whether to use, or how to use, programmed instruction. The only sure way to find out whether or not programmed instruction can be beneficial is to try out, improve and evaluate programs in a typical educational setting.

Administrators should assume the responsibility for the introduction and evaluation of programmed instruction, just as they do for all other kinds of instructional innovations. A cautious approach is important when any school considers committing its curriculum, or any part of it, to programmed learning. Careful attention must be paid to preparing the teaching staff, preparing the students, assessing the costs, providing administrative support, and selecting and preparing programs.

The preferred way to introduce programming into a school is to encourage an interested, informed, capable teacher to try a program on a limited basis. Programs covering only one topic can be tried without disrupting current school practices. For the informed teacher with some experience, programmed material covering an entire course or term can be used. Of course, it should be obvious that programmed material must be well organized and well written to be useful. Teachers who would not use a second-rate textbook should not feel that poor quality programmed material is suitable for student usage.

All programs designed for classroom use need to be supplemented. They are not materials designed to replace teachers, to avoid the need for discussions, or to replace all contact between students and professors. The lecture is not dead, discussion is not dead, classroom contact is not dead, and those few who make such claims invariably do so by lecturing to their audience without seeming to recognize the contradiction.

PREPARING TEACHERS TO USE PROGRAMMED INSTRUCTION

The reaction of teachers to programmed instruction is varied. Some regard programming as useless, or as a means to water down the rigorous courses in the curriculum. Others view the introduction of programmed material as a device to cut down on their student contact hours (reduce their teaching load); still others respond with unbounded enthusiasm and view programming as a method that will revolutionize education. Most knowledgeable instructors experienced in programmed instruction operate somewhere between the extremes. Some teachers are favorably impressed but others respond negatively to programming. Why? Perhaps because programs are new, and published programmed texts on engineering subjects are scarce. Teachers are untrained in the use of programs, schools are not geared to their use, particularly when they are used in self-paced ways, and some faculty members are openly hostile to departure from traditional practice. In any event, there are two key factors in the successful use of programmed instruction. One key is the programmed material; the other key is the teacher.

Provision should be made for preparing the teaching staff to use programmed materials. The minimum training that teachers need must produce an understanding to the basic learning theory relative to programming, the role that specific objectives play in program development, the sequencing of content, the basic principles of frame writing, the uses to be made of student responses, the interpretation of test data, and the correlation of programs with other instructional activities.

Teachers should continue to be the directors and the skilled guides to the learning process. Programs are no threat to a teacher's existence. Instead, they are a tool which, if properly prepared and utilized, can help students reach selected educational objectives efficiently and effectively. Teachers and students will enjoy using programs which challenge their competence and, at the same time, aid in effective learning.

Teachers who will be required to use programs should have the opportunity to develop a basic understanding of programmed learning. The opportunity can be provided by enrollment in courses, seminars, workshops, or intensive study of selected written materials. It is a mistake to assign teachers with no prior knowledge or experience with programmed methods to teach programmed courses.

Teachers should be capable of taking a body of programmed material and selecting from it programs for specific purposes. Most teachers do not hesitate to select appropriate material in textbooks for student assignment; yet many who are faced with programmed material prepared by others seem to think that all of it must be assigned to the students "because it is there." Attitudes such as this reflect not only inexperience but also relate to a basic lack of knowledge on the part of the teacher.

Some students react negatively to programs and do poorly with them. It may be disinterest, a reading handicap, inability to study alone, or for reasons not understood. Usually a diversity of programmed material sufficient to appeal to these students is not available; teachers should advise them to enroll in traditional classes, if possible.

Some of the better students become bored, unless they are permitted to pace themselves in their progress through the course. When self-pacing is permitted, the superior students regard finishing the course early as a challenge, and usually take off running. Most are extremely proud and satisfied if they can finish several weeks early. Naturally, a course does not need to be programmed to be self-paced, but, on the other hand, the very nature of programmed material lends itself to self-pacing.

Since experience has clearly shown that a group of learners will vary widely in their speed of learning and progress through the course, the teacher will soon find that the students are spread through widely varying subject matter areas. This means, as a practical matter, that the teacher must know the whole subject thoroughly before the course begins. This feature of programmed instruction will frighten teachers who try to stay only one day ahead of their students.

In addition, the teacher will need a considerable depth of understanding in the subject matter. Because the better students go faster, they will need more subject matter, harder problems, or higher level subjects, particularly if self-pacing through the course is not permitted. Teachers with deficiencies in subject matter, or who lack confidence in their ability to handle new and different situations, should not attempt programmed instruction.

Teachers need to be quick to recognize those students who are not progressing satisfactorily, and to attempt to find a solution to the problem. As in any kind of teaching situation, it is easier to detect the existence of a problem than it is to do something positive about it. Students may complain of boredom, lack of understanding of the frames, and of excessive slowness. They may be unable to concentrate and they may fail the tests. The failure to learn may be due to causes within the learner, the program or the teacher. Programmed learning needs the support of continued and frequent testing, with aids and procedures for remedial learning.

PROGRAMS AS HOMEWORK

Programmed material designed for use in a university setting is designed primarily for students' use at a time and place chosen by the student. In other words, programs are designed for use as homework. Failure to recognize this fundamental fact has led some to predict that teachers will become learning resource managers, whatever those are. Teachers in engineering who use programmed material will need to continue doing what they have always done--clarify concepts, provide motivation, monitor student progress and evaluate the results.

The use of programmed material, properly prepared and utilized, increases the student's efficiency in doing his homework, and makes it possible for him to get more information on his own. That is all!

If a teacher recognizes when and where programmed material is being used and for what it is being used, he can view the situation in its proper perspective. Traditionally, homework assignments have frustrated learners, encouraged copying, developed in students a dislike for school, and created antagonism among students, parents and teachers. The careful construction of programmed frames, with the immediate reinforcement provided for the successful accomplishment of small tasks, lends itself to the development of better attitudes about homework, besides increasing the efficiency of the process.

Good homework should be individualized, require independent study, relate directly to the course objectives, challenge the understanding and skill already possessed by the learner, and motivate the students to learn.

In programming, we find a method which allows individual progress at a rate controlled by the learner. We find that independent study is required by the nature of the material; it would be impractical to have someone else respond to the frames. In addition, the programs are designed to produce successful responses rather than unsuccessful ones, removing some of the motivation to cheat. If self-pacing in the class is permitted, the varying abilities of the students soon result in a considerable scatter over the length of the course, making independent study the practical way to get the work done.

Programs by definition are carefully designed to produce terminal behaviors that are clearly associated with course objectives. Through careful construction of programs the material relates directly to course objectives, thereby eliminating irrelevant material and unnecessary duplication.

In programming, we find material that should challenge the understandings and skills already possessed by the learners. The students should be required to think, to work up to their abilities, and to use as many of the skills they possess as is possible or reasonable. Programs, because of their careful sequencing and active responses, keep the learner steadily progressing and achieving. The student is continually being challenged by new material. The danger of the learner getting into problems beyond his grasp is reduced because of the small steps, small information increments, and the logical sequencing of the material.

Programmed material can motivate the learner favorably. Homework must compete with television, movies, games, sleep and other distractions. Homework assignments should be stimulating and worthwhile. Programs help to meet these requirements, and provide success which encourages progress.

The use of programmed material permits the student a more effective means of acquiring new basic information and understanding. The teacher has more class time to clarify concepts, to provide more depth, and to increase the effectiveness of the class in problem-solving techniques.

Many administrators and teachers are concerned with the extent to which a curriculum should be programmed. The answer is to use programs whenever their effectiveness can be demonstrated. The first step in finding out whether or not a program will work is to try one.

PROGRAMMED INSTRUCTION IN MECHANICS AT GEORGIA TECH

During the past few years the innovations in teaching methods employed in engineering mechanics courses at Georgia Tech have consisted of the use of programmed textual material in self-study courses. The particular courses that are currently utilizing programmed material and being conducted on a self-study basis are the undergraduate courses in statics and dynamics.

Dynamics

The possibility of teaching mechanics courses by use of programmed instruction material had been seriously considered for several years. The first attempt was made in the area of undergraduate dynamics. Since the summer of 1970, one or two sections of dynamics have been taught every quarter from a programmed text. The text used was A Programmed Introduction to Dynamics by Clyde E. Work; it was commercially available from McGraw-Hill Book Company. Other sections of the same course are still taught using a conventional text and traditional teaching methods. In this way it has been possible to compare the performance of students in the regular courses.

The instructor has been granted freedom by the administration to experiment with various ways of conducting the course. As a result, the operation of the course has proceeded from one extreme to the other. At the one extreme there are no formal class sessions; the students come to the instructor's office for individual assistance when needed. At the other extreme there is a fairly rigid schedule of lectures and help sessions. Experiments with no formal lectures at all have not proved satisfactory. Many students fail to grasp the concepts readily on their own, but yet do not seek the instructor's help probably because they fail to realize they need assistance until after the exams. On the other hand, the schedule of frequent lectures with the programmed text serving much the same function as a nonprogrammed text fails to take full advantage of the programmed material, in that programmed material should allow the student to understand the concepts and problems with less assistance than is normally required in a course utilizing a conventional text. As a result of experimentation with various systems it has been found that for the five-hour, one-quarter course in dynamics the instructor needs to meet the class formally twice each week in order to clarify concepts, to discuss and illustrate problem-solving techniques, and to provide motivation.

The student response to this innovation in teaching dynamics has been favorable as is evidenced by what they tell the faculty, by their performance in the course, any by what they tell their friends. The five-hour, one quarter course in dynamics is no longer required in any engineering curriculum, but many students are continuing to take dynamics in this fashion either in place of the current three-hour, one quarter required course, or as an elective. For example, in the spring quarter of 1973 there were 62 students enrolled in the course, even though their curricula required only a three-hour dynamics course for credit in this area of study. The programmed sections regularly reach capacity early in the registration period.

Student performance on examinations has been similar to that of other sections conducted using nonprogrammed texts with a regular schedule of five lectures per week. A typical class averages about 2.4 on a 4.0 grade scale either way. Students have claimed that the programmed course requires less work than the regular course. It is expected that this would be the case since, in the regular course, much of the student's time is wasted in the instructor's attempt to simultaneously satisfy the diverse needs of a variety of students. In the programmed course the individual can easily adapt the course to fill his individual needs.

Because of the obvious saving in instructor time, more students can be handled; this also pleases the administration. The student-credit-hour load of the instructor teaching programmed dynamics is consistently among the higher ones in the department (about 25% higher than before).

Statics

Until quite recently no programmed text for statics comparable to the programmed dynamics text cited earlier was available from a commercial publisher. Thus, the initiation of programmed instruction in statics was necessarily delayed. However, favorable experience with the dynamics course led to the decision early in 1971 to develop Georgia Tech's own statics text using a programmed format. Three faculty members financed by Institute and NSF funds devoted the equivalent of about one man year to the development of suitable material. The result was a programmed statics text in two volumes, printed on campus, for use beginning with the fall quarter of 1971.

Five sections with a total of 142 students were taught from the programmed material during the 1971 fall quarter; more than 350 other students were enrolled in statics sections which were being taught in the traditional way, using Vector Mechanics for Engineers by Beer and Johnston as the text. The faculty members teaching the programmed sections were free to administer the sections as they chose and were encouraged to experiment with innovations of any reasonable kind in the administration of the course. It was decided to divide the material in four main parts and to have an exam on each of these parts. The division was as follows:

1. Moments, forces, free-body diagrams, equivalent force systems.
2. Centroids and centers of mass, equilibrium concepts.
3. Beams, trusses, frames and machines.
4. Equilibrium in three dimensions, friction.

Seven dates for giving examinations to students in all sections as a group were chosen. Any student who wanted to repeat an exam was permitted to do so, within the limits imposed by the seven dates.

As a result of this effort, student performance was somewhat higher in the programmed sections than in those using a conventional method. This was due to the self-pacing features of the programmed instruction sections, which allowed time to repeat examinations in a limited number of cases until the material was mastered.

During the winter quarter of 1972 five programmed sections were offered under four different instructors. During registration, a fifth section was added to the four originally scheduled but, even so, several dozen students who asked to enroll in the programmed statics were turned down due to lack of space. The final figures showed 169 in the programmed sections and 301 in the sections which used traditional methods. A group final examination was scheduled at the close of the winter quarter for all statics sections. The performance of all sections was similar. For the spring quarter, enrollment in the programmed sections was cut off when the balance of the 500 texts which had been printed for use in 1971-72 was committed at registration.

A few of the observations made during the conduct of the programmed sections in 1971 will now be discussed. First, as might be expected, there was a general tendency on the part of the student when required to budget his own time to get behind in the course and then try to do too much during the last days of the quarter. Also, without the urgency of day-by-day assignments, students tended

to study statics only when the demands of their other courses had been met, and in many cases this amounted to a few days of concentrated study immediately prior to taking an examination. Secondly, with the course divided into four parts, a student could deceive himself into thinking that he understood one of these parts quite well and then find out otherwise as a result of the examination. In many cases, by that time the student was sufficiently behind that it was very difficult to take action to remedy the situation, since he had to not only go back and repeat the material he had failed but at the same time proceed forward into the new material.

Based on the experience with using the text during the 1971 school year, it was decided during the summer of 1972 to revise the text rather substantially, clarifying some of the more difficult concepts, adding more problems where needed, correcting errors in the original text, and making other changes to more properly adapt the text to fulfill its objectives.

Prior to the fall quarter of 1972 several changes were made in the administration of the programmed sections. The course was divided into ten parts, each part corresponding to a unit in the text. The division was as follows:

1. Vectors
2. Forces and moments
3. Equivalent force systems
4. Centroids and centers of mass
5. Equilibrium concepts
6. Frames and machines
7. Trusses
8. Beams
9. Equilibrium in three dimensions
10. Friction

A one-half hour exam was given on each of these ten topics, with the student being required to pass all ten of these basic exams in order to obtain a C in the course. Failing to pass one of the basic exams resulted in a D grade; failing to pass two resulted in not passing the course. In addition, supplementary exams were given on wrenches and belt friction, and advanced exams were given in the areas of centroids and centers of mass, frames and machines, trusses, beams, equilibrium in three dimensions, and friction. After satisfying the requirements for a C, a student could obtain either a B or an A by passing two or four supplementary or advanced exams, respectively. During the quarter the student was given twenty opportunities to take exams. The classes met once a week, during which time some of

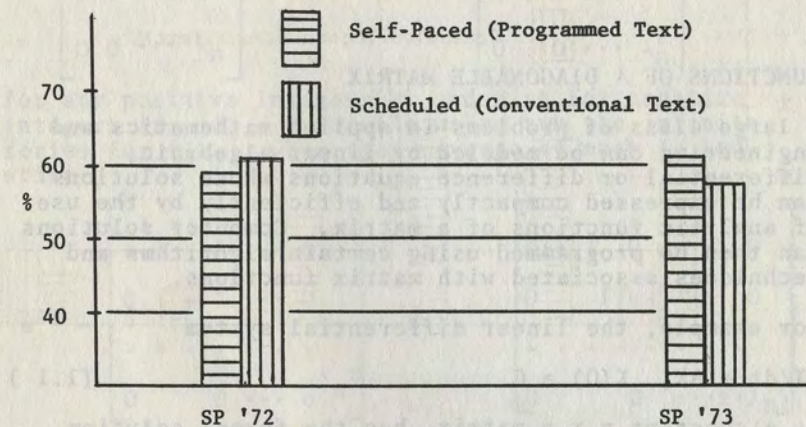
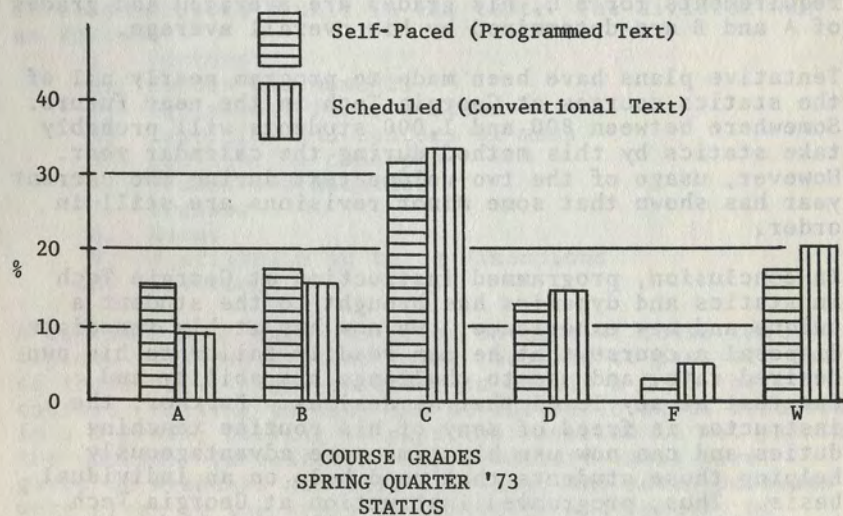
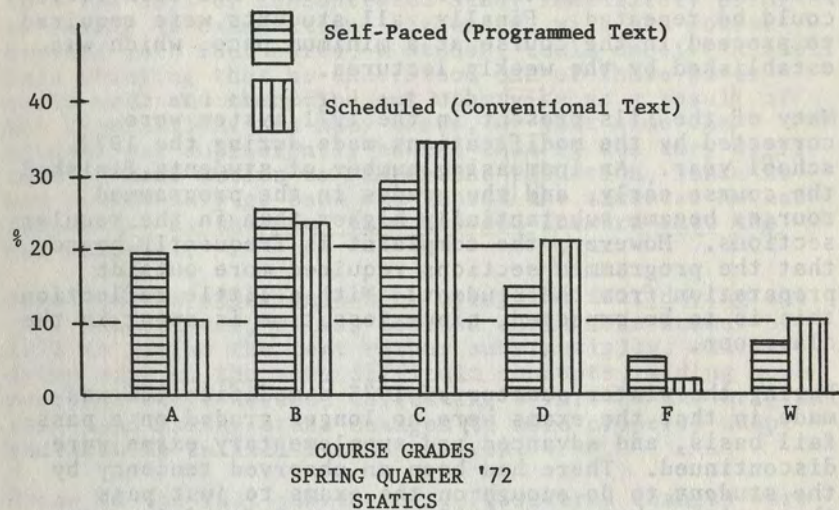
the more important or more difficult concepts were discussed and techniques to solve various types of problems were demonstrated. Tests that had been failed could be repeated. Finally, all students were required to proceed in the course at a minimum pace, which was established by the weekly lectures.

Many of the ills present in the 1971 system were corrected by the modifications made during the 1972 school year. An increasing number of students finished the course early, and the grades in the programmed courses became substantially higher than in the regular sections. However, the complaint is frequently heard that the programmed sections required more outside preparation from the student. With a little reflection this is to be expected, since less time is spent in the classroom.

During the winter quarter of 1973 a modification was made in that the exams were no longer graded on a pass-fail basis, and advanced and supplementary exams were discontinued. There had been an observed tendency by the student to do enough on the exams to just pass them, since there was really no reward for doing any better. Now, after the student has satisfied the requirements for a C, his grades are averaged and grades of A and B are determined by his overall average.

Tentative plans have been made to program nearly all of the statics courses at Georgia Tech in the near future. Somewhere between 800 and 1,000 students will probably take statics by this method during the calendar year. However, usage of the two-volume text during the current year has shown that some minor revisions are still in order.

In conclusion, programmed instruction at Georgia Tech in statics and dynamics has brought to the student a unique and new experience. He now has at his immediate disposal a course that he can readily tailor to his own desired rate, and use to challenge his ability and interest at any level that he desires. Further, the instructor is freed of many of his routine teaching duties and can now use his time more advantageously helping those students that need help on an individual basis. Thus, programmed instruction at Georgia Tech has helped students to more readily relate their study to the achievement of the course objectives.



APPLICATIONS OF MATRIX FUNCTIONS

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FUNCTIONS OF A DIAGONABLE MATRIX

A large class of problems in applied mathematics and engineering can be modeled by linear algebraic, differential or difference equations whose solutions can be expressed compactly and efficiently by the use of analytic functions of a matrix. Computer solutions can then be programmed using certain algorithms and techniques associated with matrix functions.

For example, the linear differential system

$$dX/dt = AX, \quad X(0) = C \quad (1.1)$$

A, a constant $n \times n$ matrix, has the formal solution

$X = e^{At}C$, and the system

$$d^2X/dt^2 + B^2X = 0, \quad X(0) = C, \quad (dX/dt)_0 = V \quad (1.2)$$

has the formal solution

$$X = (\cos Bt)C + (B^{-1} \sin Bt)V. \quad (1.3)$$

Here the functions of matrices A and B^2 may be defined by series

$$e^{At} = I + At + A^2t^2/2! + \dots \quad (1.4)$$

$$\cos Bt = I - B^2t^2/2! + B^4t^4/4! - \dots \quad (1.5)$$

$$B^{-1} \sin Bt = It - B^2t^3/3! + B^4t^5/5! - \dots \quad (1.6)$$

so that only positive integral powers of A and B^2 are involved in the solution. There are important computational advantages, however, in expressing these and other analytic matrix functions (defined by polynomials or convergent series in A or A^{-1}), in finite form as polynomials of degree less than n in the given constant matrices. This is easily done if the given matrix is a diagonal matrix Λ , with diagonal entries λ_j called its eigenvalues. For the equations

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, \quad \Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}, \quad (1.7)$$

for any positive integers k, and also for negative integers if $|\Lambda| \neq 0$, enable us to sum the infinite series for $f(\lambda_j)$, when they converge at each λ_j and write

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}, \quad f(\Lambda) = \begin{bmatrix} f(\lambda_1) & 0 & \dots & 0 \\ 0 & f(\lambda_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f(\lambda_n) \end{bmatrix}. \quad (1.8)$$

Suppose that Λ has s distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$, where λ_j appears with multiplicity m_j and $\sum m_j = n$. Then the Lagrange interpolating polynomial defined by

$$q_j(\lambda) = \prod_{\substack{h=1 \\ h \neq j}}^s (\lambda - \lambda_h) / (\lambda_j - \lambda_h) \quad (1.9)$$

has the property that $q_j(\lambda_k) = \delta_{jk}$. Hence, the matrix function

$$E_j = q_j(\Lambda) = \begin{bmatrix} q_j(\lambda_1) & & 0 \\ & q_j(\lambda_j) & \\ 0 & & q_j(\lambda_n) \end{bmatrix} \quad (1.10)$$

is a diagonal matrix having m_j 1's in the entries occupied by λ_j in Λ , and 0's elsewhere. We find

$$\sum_{j=1}^s E_j = I \quad \text{and} \quad (1.11)$$

$$E_j \cdot E_k = E_j \delta_{jk}, \quad (1.12)$$

so the E_j are called mutually orthogonal idempotent matrices. Since the E_j are polynomials in Λ that satisfy the equations

show that for $\lambda = \lambda_1 = 4$, the equation $(4I - A)S_1 = 0$ has one solution vector, whereas for $\lambda = \lambda_2 = 1$, the rank of $I - A$ is 1, and there are $3 - 1 = 2$ independent solutions S_2, S_3 such that $[I - A][S_2, S_3] = 0$. The modal equation $AS = SA$ becomes

$$AS = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = SA. \quad (1.29)$$

A normal matrix is an $n \times n$ (real or complex) matrix A such that

$$A^*A = AA^*, \quad (1.30)$$

where A^* denotes the complex conjugate of the transpose of A . The class of normal matrices includes hermitian (including real symmetric) matrices, such that $A^* = A$, skew hermitian (including real skew) matrices, such that $A^* = -A$, and unitary (including real orthogonal) matrices, such that $A^* = A^{-1}$. Their eigenvalues satisfy respectively

$$\bar{\lambda}_j = \lambda_j, \text{ real, for hermitian matrices,} \quad (1.31)$$

$$\bar{\lambda}_j = -\lambda_j, \text{ pure imaginary, for skew hermitian matrices,}$$

$$\text{and } \bar{\lambda}_j = \lambda_j^{-1}, \text{ of absolute value 1, for unitary matrices.}$$

Every normal matrix is diagonalizable by a unitary modal matrix S , from whose columns we can form the mutually orthogonal idempotents $S_j S_j^*$ of rank 1. For a root λ_j of $D(\lambda) = 0$ of multiplicity $m_j > 1$, we must add m_j such idempotents to obtain the constituent idempotent A_j . If S_j is an eigenvector of A , but is not necessarily of length 1, the idempotent is

$$A_j = \sum S_k S_k^* / (S_k^* S_k), \text{ summed for } \lambda_k = \lambda_j. \quad (1.32)$$

Given a diagonal matrix A_j we can expand certain known functions $f(\lambda) = 1, \lambda, \lambda^2, \dots, \lambda^{s-1}$ in terms of the unknown idempotents, and then solve for $A_1, A_2, \dots, A_s$ in terms of the known powers $I, A, A^2, \dots, A^{s-1}$. In the example above with $\lambda_1 = 4, \lambda_2 = 1$, and A in (1.26) we have

$$\text{for } f(\lambda) = 1 : I = 1 \cdot A_1 + 1 \cdot A_2, \text{ and} \quad (1.33)$$

$$\text{for } f(\lambda) = \lambda : A = 4 A_1 + 1 \cdot A_2. \quad (1.34)$$

Hence,

$$A_1 = \frac{A - I}{4 - 1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_2 = \frac{A - 4I}{1 - 4} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad (1.35)$$

APPLICATIONS TO LINEAR STATE MODELS

A common problem which arises in network theory and in more general types of linear systems is the solution of the equation

$$\frac{dX}{dt} = AX + BU(t), \quad (2.1)$$

where X is an unknown $n \times 1$ vector function of time, $U(t)$ is a given $m \times 1$ "control" vector function of time, and A and B are constant n -rowed matrices of n and m columns, respectively.

If A is diagonalizable with modal matrix S and diagonal spectral matrix Λ , then the change of variables $X = SY$ reduces equation (2.1) to

$$S \frac{dY}{dt} = ASY + BU(t). \quad (2.2)$$

Left multiplication by $R = S^{-1}$ yields the simplified system

$$\frac{dY}{dt} = \Lambda Y + RBU(t) \quad \text{or} \quad \frac{dy_j}{dt} = \lambda_j y_j + R_j BU(t). \quad (2.3)$$

Thus, the variables are separated, and the explicit solution is

$$y_j(t) = e^{\lambda_j t} y_j(0) + \int_0^t e^{\lambda_j(t-\tau)} R_j BU(\tau) d\tau. \quad (2.4)$$

As t increases, the leading term, obtained for $B = 0$, will tend to 0, oscillate, or become infinite, accordingly, as the real part of the eigenvalue λ_j is negative, zero, or positive. The linear system (2.1) is called stable if none of the eigenvalues λ_j of A have positive real part, and no multiple eigenvalue is purely imaginary.

The solution to (2.1) can be expressed directly in terms of A , using functions of A , in the form

$$X = e^{At}X(0) + \int_0^t e^{A(t-\tau)}BU(\tau)d\tau, \quad (2.5)$$

if formula (1.20) is used to express the function e^{At} in terms of the function values $e^{\lambda_j t}$ and the idempotents $A_j = S_j R_j$.

APPLICATIONS TO MECHANICAL VIBRATIONS

The differential system

$$M \frac{d^2 X}{dt^2} + KX = 0, \quad X(0) = C, \quad \left. \frac{dX}{dt} \right|_0 = V \quad (3.1)$$

furnishes a mathematical model for a variety of mechanical vibration problems, such as the vibration of airplane wings or of multistoried buildings, where the coordinates x_j may represent displacements of a sequence of points that are equally spaced at equilibrium.

A simple example concerns the longitudinal oscillations of n equal masses attached to a uniform horizontally stretched spring of length $L = (n+1)h$, at distances kh from one of the fixed ends when the spring is at rest. If x_i denotes the subsequent horizontal displacement of the i th mass from equilibrium, and k^2_m is the spring constant, then by Hooke's Law we have

$$\frac{m d^2 X}{dt^2} = mk^2 \cdot \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \cdot X. \quad (3.2)$$

This equation may be rewritten in the form

$$\frac{d^2 X}{dt^2} + k^2 [2I - A]X = 0 \quad (3.3)$$

and be solved by (1.3), where $B^2 = k^2(2I - A)$, and A is the matrix (3.5) with entries 1 immediately above and below the diagonal and 0's elsewhere. Now the trigonometric identity

$$\sin(i-1)j\theta + \sin(i+1)j\theta = (2 \cos j\theta) \sin i(j\theta) \quad (3.4)$$

shows that $A S_j$ and $S_j \lambda_j$ will have the same i th entry if

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad S_j = \begin{bmatrix} \sin j\theta \\ \sin 2j\theta \\ \sin 3j\theta \\ \vdots \\ \sin nj\theta \end{bmatrix}$$

$$\lambda_j = 2 \cos j\theta, \quad \theta = \frac{\pi}{n+1}. \quad (3.5)$$

Since A is real symmetric, it is normal, and its constituent idempotents A_j have the form $S_j S_j^* / (S_j^* S_j)$.

The squared length of S_j is

$$\begin{aligned} S_j^* S_j &= \sum_{i=1}^n \sin^2 i j \theta = \frac{1}{2} \sum_{i=0}^n (1 - \cos(2ij\theta)) \\ &= \frac{n+1}{2} - \frac{1}{2} \sum_{i=0}^n \cos 2ij\pi/(n+1) = \frac{n+1}{2} - 0. \end{aligned} \quad (3.6)$$

Thus, A has the constituent idempotents

$$A_j = \frac{2}{n+1} S_j S_j^*. \quad (3.7)$$

To compute the eigenvalues μ_j of $B = k(2I - A)^{1/2}$, we replace A by its eigenvalue $\lambda_j = 2 \cos j\theta$, and find

$$\mu_j = k(2 - 2 \cos j\theta)^{1/2} = 2k \sin j\varphi, \quad (3.8)$$

where $\varphi = \theta/2 = \pi/2(n+1)$.

Finally, using (1.3), we obtain the explicit solution of (3.3):

$$X = \sum_{j=1}^n [(S_j^* C) \cos(2kt \sin j\varphi) + (S_j^* V) \frac{\sin(2ktsin j\varphi)}{2k \sin j\varphi}] \frac{2S_j}{n+1}. \quad (3.9)$$

If the more general equation (3.1) is to be studied for large values of n , it may suffice for practical purposes to find only the lowest oscillation frequencies $w_1, w_2, \dots$ with their corresponding modal vectors S_j .

A vector $e^{i\omega t} S_j$ satisfies the differential equations (3.1) if and only if

$$(-w^2 M + K) S_j = 0, \text{ or } (w^{-2} I - K^{-1} M) S_j = 0. \quad (3.10)$$

The numbers $\lambda_k = w_k^{-2}$ represent for $k = 1, 2, 3$ the largest eigenvalues of the matrix $A = K^{-1} M$, for which S_k is a corresponding eigenvector. Since the symmetric matrix $M^{1/2} K^{-1} M^{1/2}$ has the eigenvector $M^{1/2} S_j$, it has the corresponding idempotent

$$M^{1/2} S_j S_j^* M^{1/2} / (S_j^* M S_j).$$

Thus, the idempotents of A are

$$S_j S_j^* M / (S_j^* M S_j). \quad (3.11)$$

If the largest eigenvalue λ_1 has multiplicity 1, we can calculate it without finding and solving the characteristic equation by using the following iteration procedure: Starting with the first column I_1 of the unit matrix, we successively multiply by A and divide by the first non-zero coefficient. Then the limiting vector is the eigenvector S_1 , and the limiting divisor is the largest eigenvalue λ_1 . From (1.20) we have

$$(A/\lambda_1)^k I_1 = [A_1 + (\lambda_2/\lambda_1)^k A_2 + \dots + (\lambda_n/\lambda_1)^k A_n] I_1, \quad (3.12)$$

and for large k the right hand side approaches $A_1 I_1$, which is an eigenvector that is unaffected by multiplication by A/λ_1 .

A DIFFUSION PROBLEM IN FLUID FLOW

In a certain chemical engineering process a fluid flows continuously with constant velocity C in the positive s direction through a long cylindrical tube or reactor of length L with constant circular cross-section. At time $t = 0$, a new fluid is introduced at the input end and flows through with the same velocity. Part of the old fluid is washed out, but part diffuses back through the new fluid. The problem is to determine the concentration $x(s, t)$ of the new fluid as a function of positive s and time t .

We first assume that the variation of x can be described by the Taylor diffusion model

$$\frac{\partial x}{\partial t} = -C \frac{\partial x}{\partial s} + D \frac{\partial^2 x}{\partial s^2} \quad (4.1)$$

with velocity C and dispersion coefficient D , subject to the initial and boundary conditions

$$x(s, 0) = 1 \text{ for } 0 \leq s \leq L, \text{ and} \quad (4.2)$$

$$x(0, t) = 0, \quad \frac{\partial^2 x(L, t)}{\partial s^2} = 0, \text{ for } t \geq 0. \quad (4.3)$$

Next we approximate the partial derivatives with respect to s by finite differences, taking $s = kh$, $h = L/n$, and obtain a system of first-order ordinary differential equations

$$\frac{dx_k}{dt} = -C \left(\frac{x_{k+1} - x_{k-1}}{2h} \right) + D \left(\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} \right), \quad (4.4)$$

where $x_k(t) = x(kh, t) = 1$ for $t = 0$, and

$$x_0 = 0, \quad x_{n+1} - 2x_n + x_{n-1} = 0 \text{ for } t \geq 0. \quad (4.5)$$

With a new time variable τ and parameter p defined by

$$\tau = tD/L^2, \quad p = Ch/2D = \sin \varphi, \quad (4.6)$$

the differential equation (4.4) becomes

$$n^{-2} \frac{dx_k}{d\tau} = (1+p) x_{k-1} - 2x_k + (1-p) x_{k+1}. \quad (4.7)$$

We eliminate x_0 and x_{n+1} by the boundary conditions (4.5), introduce the n -vector X with components $x_1, \dots, x_n$ and combine equations (4.5) into a single matrix equation:

$$n^{-2} \frac{dX}{d\tau} = -MX. \quad (4.8)$$

The tridiagonal coefficient matrix $-M$ is not symmetric, but may be made symmetric by transforming by a diagonal matrix B . We define

$$q = (1-p^2)^{1/2} = \cos \varphi, \quad b = (1-p)/q = \tan(\pi/4 - \varphi/2), \\ c = (1-b^2)^{1/2}, \quad (4.9)$$

$$M = 2I - q B^{-1}AB, B = \text{diag}\{b, b^2, \dots, b^{n-1}, b^n/c\}, \quad (4.10)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 & 0 & c \\ 0 & 0 & \dots & 0 & c & 2b \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \quad (4.11)$$

The required solution of the differential equation that satisfies the initial and boundary conditions may be written

$$X = e^{-Mn^2\tau} X_0. \quad (4.12)$$

To the eigenvalue $\lambda_j = 2 \cos \theta_j$ of the symmetric matrix, A corresponds the eigenvalue

$$\mu_j = 2 - q\lambda_j = 2(1 - \cos \varphi \cos \theta_j) \quad (4.13)$$

of M . The eigenvector S_j of A in (3.5) must be modified slightly to give the following eigenvector for A in (4.11):

$$S_j^* = [\sin \theta_j, \sin 2\theta_j, \dots, \sin(n-1)\theta_j, (\sin n \theta_j)/c] \quad (4.14)$$

where θ_j is not $j\pi/(n+1)$ as before, but is determined by an equation derived from the last row of A , namely

$$c \sin(n-1)\theta_j + 2(b/c) \sin n \theta_j = (2 \cos \theta_j \sin n \theta_j)/c \quad (4.15)$$

which may be rewritten as

$$\sin(n+1)\theta_j - 2b \sin n \theta_j + b^2 \sin(n-1)\theta_j = 0. \quad (4.16)$$

The constituent matrices for A and M are

$$A_j = S_j S_j^* / (S_j^* S_j) \text{ for } A, B^{-1} A_j B \text{ for } M, \quad (4.17)$$

where by the aid of some complicated trigonometric identities we find

$$S_j^* S_j = (n/2)(1 + v_j^{-1}), \text{ and} \quad (4.18)$$

$$v_j = \mu_j n/2p = \mu_j n^2 D/CL. \quad (4.19)$$

Restoring the time variable t from (4.6) in the solution (4.12), and using v_j from (4.19), we obtain on expanding the exponential function

$$X = \sum_{j=1}^n r_j e^{-v_j Ct/L} B^{-1} S_j, \quad (4.20)$$

where r_j denotes the scalar quantity

$$r_j = S_j^* B X_0 / (S_j^* S_j), \quad (4.21)$$

which can be reduced by a series of trigonometric identities to

$$r_j = (\sin \theta_j) / (\tan \varphi)(1 + v_j). \quad (4.22)$$

To obtain $\lambda_j = 2 \cos \theta_j$, and hence μ_j and v_j , we must solve the characteristic equation (4.16). We observe that the left side of (4.16) is the imaginary part of $zn(z^{1/2} - bz^{-1/2})^2$ for $z = e^{i\theta}$. Noting that $\cot(\varphi/2) = (1+b)/(1-b)$, we can express condition (4.16) in the form

$$j\pi = n \theta_j + 2 \tan^{-1}(\cot(\varphi/2) \tan(\theta_j/2)). \quad (4.23)$$

We determine $\varphi = \sin^{-1}(Ch/2D)$ from (4.6), solve for θ_j by iteration in (4.23), calculate $v_j = (1 - \cos \varphi \cos \theta_j) \cdot n/\sin \varphi$, find r_j from (4.22) and compute the solution X from (4.20). Then

$$x_k = \sum_{j=1}^n e^{-v_j Ct/L} b^{-k} \sin k \theta_j \sin \theta_j / (1 + v_j) \tan \varphi, \quad 1 \leq k \leq n. \quad (4.24)$$

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RESEARCH SYNDROMES

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I would like to say a few words about the quality of mechanics research, or more precisely of engineering research in general, starting with a true story of a technical meeting I attended about 5 years ago. This meeting was set up for the purpose of acquainting the scientific community with the latest results in upper atmosphere bomb phenomenology. It was attended by about a thousand people and lasted 3 days. Each day there were about 15 papers, and each paper employed 15 to 20 figures or tables. That is nearly a thousand figures and tables - almost entirely raw data, either experimental results or computer calculations. Nobody was allowed to take any notes for security reasons. When the meeting was over I don't believe any of us could have talked more than 10 minutes on what we had learned in the 3 days, which is a damn poor rate of return on investment.

On the morning of the second day, a paper was given on a computer analysis of a 1 megaton burst occurring at 30,000 foot altitude. There were scads of graphs giving the results, and I found myself struggling to make sense out of them. One, for instance, was a logarithmic plot of the time after burst at which the shock wave would arrive at various distances from the point of burst. I found myself using one finger to help estimate the slope of the line to see if it was $2/5$, as predicted by Taylor's simple theory of point explosions published in the Proceedings of the Royal Society about 20 years ago. But before I could quite manage it, the slide disappeared, and we were looking at the next one.

It turned out that in the break following this paper, I wound up in the coffee line behind the sponsoring agency's chief scientist. I mentioned that we were being inundated by a flood of undigested data that nobody could assimilate and take away with him. I said that as soon as I got my copy of the proceedings I would get out a ruler, check slopes, sort out which results agreed with what I expected, which didn't, and try to interpret the deviations. But instead of one thousand individuals each doing this for himself, I said, why doesn't the author do it? He said he hadn't thought about it before, but it seemed like a good idea. He would try to see that this is done in future meetings.

The very next paper was one dealing with the problem of an astronaut in a space capsule at 100 miles altitude attacked by a 1 megaton co-altitude nuclear bomb. The author calculated the neutron flux through the astronaut's body for various burst distances. He told us with some pride how he used FORTRAN to determine the neutron flux inside the space capsule, and he gave the results as a logarithmic plot of flux versus distance with about 50 computed points. They fell on a very good straight line. I checked the slope and found that it was minus 2. I then noted that the chief scientist was in the row behind me and moved over to speak to him. "Look," I said, "I believe our speaker has just proved the inverse square law!" He look surprised and then burst into a laugh which must have startled the rest of the audience. I suspect that less than 1% of those there arrived at this or any other interpretation. The reason is very simple. We are so accustomed to barrages of meaningless data that nearly all of us turn off our minds and stop thinking.

I chose this example because I think it illustrates rather well a number of syndromes which I believe afflict engineering research today, and I would like to spend the rest of my time identifying and discussing some of these.

One syndrome is the standard practice of writing reports designed to impress rather than inform. In the case of the astronaut and the space vehicle, the most meaningful way to report the results, and one that we could have carried away with us, would be to say that for the particular space capsule the fraction of neutrons that got through and hit the astronaut was $x\%$. When you come right down to it, the shielding factor (which actually wasn't mentioned) was really the entire useful content of what the investigator did. But that doesn't sound nearly as impressive as to report a computer program that probably employed a Monte Carlo approach to calculate the shielding, and demonstrate that it had been successfully employed in 40 or 50 cases.

A second syndrome which is related to the first is our habit of suppressing our intuitive processes, or at least destroying the evidence. A former president of the American Physical Society, named W. G. F. Swann, once wrote a book called the Architecture of the Universe, and I would like to read an excerpt from that. "A scientific researcher can be likened to two distinct animals. Ahead there travels a dog with keen scent and not over critical capacity who pokes his nose into every bit of intellectual garbage he can find on the wayside.

He ferrets out from every conceivable place new morsels which no dog has ever chewed before and gnaws out as much of their content as he can. And behind the dog there walks, clad in the most conventional apparel, a calm and dignified gentleman who views very critically everything the dog picks up, and makes him put it down again if it is found wanting, but pockets it in his immaculate coat if it is of worth. It is not to be forgotten however, that it is usually the dog that finds the bone. The dog's name is Intuition and his master's name is Precise Thought." In writing papers we generally eliminate all traces of the intuitive processes which lead to an important discovery, and give only formalized results. Intuition is somehow considered not quite respectable, possibly because when properly employed it often makes a laborious formal demonstration of something seem totally unnecessary.

Incidentally, in his speech following his acceptance of the Timoshenko medal a few years ago, Bill Prager reminisced about the old days in Gottingen and the contrasts between two of the leading figures there, Prandtl and von Mises. Prandtl always looked at things physically and sought to make things intuitively evident, while von Mises always insisted on rigorous mathematical analysis before he would accept any results. Both approaches are of course important to progress, but Prager expressed the view that today the pendulum has swung too far in the direction of formalism. Inventiveness is greatly enhanced by a careful cultivation of a broad intuitive grasp, corrected whenever it is in error. And for whatever it is worth, I suspect that in the annals of fluid mechanics Prandtl's name will be more illustrious than that of von Mises.

Another syndrome is our tendency to divorce our technical thinking from our everyday experience. As a result, we lose perceptiveness and suffer some degradation in the valuable faculty known as common sense, at least as applied to technology. As a simple example, take Kelvin's theorem about circulation. It is sometimes expressed as follows: Vorticity cannot be induced in a perfect fluid by forces derivable from a potential. Yet everybody who has ever poured tea in a cup or flushed a toilet knows that a whirlpool is set up. And everybody who knows Stokes' theorem relating the surface integral of the curl of a vector to the line integral around the surface knows that a whirlpool implies the existence of vorticity somewhere inside. But in most cases experts in fluid mechanics, to whom I have posed this paradox, had never previously been aware of any contradiction between their accepted technology and this everyday experience. And

in explanation, 90% have adopted the view that in teacups and toilets circulation does indeed exist, but it is either because water isn't a perfect fluid or because the forces applied aren't derivable from a potential. I think both explanations are wrong, but I'll leave it to you to come up with the correct one.

Then there are the cases where people apparently don't really want to clean up a field--that would deprive them a pleasant and secure job. Many of you have probably heard the story about a young lady who went into a department store to buy some material for a nightgown. She wanted something very soft and gauzy and the saleslady finally found just what she wanted. When asked how much was needed, she requested 20 yards. The saleslady was aghast and said, "My dear young lady, you obviously don't know much about making clothes. All you need for a nightgown is 2 1/2 yards." The sweet young thing replied, "But you don't know my husband. He's a research and development engineer, and he would much rather look for something than find it!"

A very serious syndrome is our habit of turning our thinking over to the computer. I am personally of the opinion that if the computer had reached its present stage of development in the time of Sir Isaac Newton, the science of mechanics as we know it today would not exist. The computer has such an enormous capacity for incorporating unimportant details as well as the main features of a system into an analysis, that the art of idealizing things into particles, rigid bodies, weightless strings, frictionless pulleys, and all the rest might never have developed; nor might the concepts of kinetic energy, potential energy, momentum and all the other abstractions which shape our thinking in mechanics.

An amusing example of this habit of relying entirely on computers occurred in the early days of the space program. When the second moon probe was launched (the first blew up), there was a great deal of excitement and newscasters gave reports every quarter or half hour on the probe's altitude and velocity. For several hours it was taken for granted that the probe would reach the moon, the only question being whether it would hit it or miss it. Being somewhat skeptically inclined, however, I took out my slide rule when I heard the first report on altitude and velocity about fifteen minutes after launch, calculated the potential and kinetic energies of the probe, and calculated the maximum altitude that it could reach. I found that it couldn't go even half way. For hours after that, newscasters kept reporting the probe's position and velocity with no doubts expressed about its ability

to reach the goal. But eventually the truth was out. The probe reached 80,000 miles, $1/3$ of the distance to the moon, returned, and fell in the ocean. This type of calculation takes less than 10 minutes, and it was amazing to me that when so many people in the United States were perfectly capable of doing it, nobody bothered; or if some did, they didn't communicate their results to the news media.

Another common syndrome is losing track of the fact that there is no such thing as an exact solution to a real physical or engineering problem. The only thing that permits an exact solution is some idealization of the physical or engineering problem, and from a practical point of view there is no great advantage in reducing the error in the solution of the idealization very much below the error by which the idealization fails to represent the problem itself accurately. There is really nothing new in this observation. In fact, I believe it was Aristotle who once said that it is the mark of an educated man not to expect greater accuracy than the subject permits. We characteristically lose sight of this advice, however, and carry out the idealization to as many significant figures as we can manage--very often at excessive cost and with unnecessary time delays.

After some experience with a given idealization, the idealization may assume greater importance in our minds than the problem itself. This reminds me of the story of the policeman who was on his beat early one morning about 1:15, and he saw a man in a tuxedo crawling around on his hands and knees under a street light. He went over thinking the man must be drunk and that he ought to run him in, but discovered it was a man he had known for years and rather liked. Upon inquiry, it turned out that he had lost his house keys. Being of a kindly disposition, the policeman got down on his hands and knees to help look for them. After about 10 minutes he became discouraged and asked the man where he lost them. "Oh," the man replied, "over the across the street." The policemen asked, "Then why don't you look over there?" to which the inebriate replied, "Don't be silly. It's dark over there!" I sometimes think that's the story of research. We tend to work on the things we feel we understand instead of the problem that really needs a solution.

Part of our excessive dependence on computers may come from an instinctive desire for good accuracy. But the accuracy that is needed depends on the problem, and often in the initial stages of an investigation very crude numbers are quite adequate. For instance, in the early days of the communications satellite program, some inventive people at Lincoln Laboratory and Rand

Corporation suggested that a very effective, practically indestructible passive communications satellite system allowing instant communication with other continents would be to have a cloud of tiny dipoles traveling in orbit and filling a complete circular path around the earth something like Saturn's rings. In order to get as many dipole reflectors into orbit as possible subject to a given weight limitation, the individual elements must be very thin. As a result, the radiation pressure of the sun becomes important in perturbing the orbital path.

At that time I was serving in the Pentagon as Technical Director of the Military Communication Satellites Program, and was very anxious to get an answer to the longevity of the system. On a Thursday afternoon, I remember calling some people who were calculating the orbital perturbation, and learned that they were putting the finishing touches on their computer program and expecting some answers the following Tuesday. Unwilling to wait, I idealized the problem as follows: The largest perturbation will come when the sun is in the orbital plane of the needles. When that happens we can divide the nominally circular orbit into four arcs. In one of them a needle is in the earth's shadow and is thus unperturbed. In the corresponding arc on the sunny side the average velocity of the needle is at right angles to the line joining the sun and the earth. The two remaining arcs consist of one in which the average direction is directly away from the sun. We can thus approximate the perturbing effect of the earth's sun radiation pressure by calculating the total impulse that will be given as a single pulse at the center of each of the arcs. It is easy to show that each of the three impulses has the effect of changing the orbit to an ellipse with its major axis at right angles to the line of joining the earth and sun, and that the ellipticity increases with each pulse. Thus the apogee increases and the perigee decreases until finally the orbit dips down into the earth's atmosphere, after which, of course, the remaining lifetime is very short. I worked out a formula for this on Friday afternoon, and was pleased to find the following Tuesday that the computer verified my results. Here was a case in which the inaccuracy involved in replacing continuous pressure by three impulses was far less than the uncertainty in the sun's radiation pressure on each dipole due to its random orientation.

By now many in the audience will probably feel I am anticomputer. Actually I'm not. The computer is of inestimable value in solving many important types of problems. For some it is even indispensable. But it

should be kept in its proper place, and must not be allowed to replace physical thinking, intuitive insight and a broad grasp of fundamentals. There is a considerable threat to these arising from analytical solutions also. When solutions are presented as complex integrals or complicated functions of variables which have been so transformed that you can't see clearly how they are related to the actual physical parameters, the results are little better than cooking recipes. Like computers, such solutions have a place, but they leave the essential job of conveying understanding still to be accomplished.

What I do think is that a complicated and expensive solution should only be resorted to when the situation warrants it. Further I believe that each such solution should be preceded whenever possible by a simple solution which scopes the problem, conveys a basic understanding of its main features, and shows whether and for what values of the parameters a more accurate solution is worthwhile. There are some people who have developed a remarkable capacity for stripping a problem down to its essential elements and getting simple approximate solutions. Most of the people I know on the President's Scientific Advisory Committee and high level defense technical councils are back-of-the envelope artists in their own right. I don't believe this is a mere coincidence; rather, there is a cause-and-effect relationship between technical competence and breadth on the one hand, and the practice of viewing a problem in its simplest terms on the other.

This is far from a complete discussion of research syndromes, but I have gone on long enough. If we could lick those I've mentioned, I believe it would improve our performance remarkably. So let me just close by summarizing.

We should carefully cultivate our basic physical understanding of the phenomena we deal with, and recognize the important role of intuition in discovery. Technical papers should be written so as to inform rather than impress. We should integrate our daily experience and our technical thinking for the additional insight and practical common sense that can result. The technical approach employed should be properly tailored to the needs of the occasion, using simple approximate methods to scope a problem and achieve a basic understanding, followed by relatively complex and expensive solutions whenever the situation warrants it. Finally, we should be guided in our choice of problems by their importance, whether theoretical or practical, and not be shackled by either our past history or tradition. If we can really follow all of these admonitions, I believe it might possibly result in something of a renaissance in mechanics.

APPLICATION OF MATLAN TO STRUCTURAL ANALYSIS

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INTRODUCTION

It has long been recognized that the computer is an indispensable tool in the stress and deformation analysis of all but the most simple structural configurations. Continued development of digital computers has made the current generation faster, more reliable, and more economical to operate than ever before. However, the primary difficulty still remains--communication with the computer. In recent years increased attention has been given to the development of problem-oriented and procedural languages in order to alleviate this difficulty (Fenves 1969; Murphree, Melin and Fenves 1966; Melin 1966). In particular, problem-oriented languages (Hart and Hargraves 1960; Miller 1961; Fenves et al. 1964) simplify greatly the task of the programmer. However, this simplification is achieved by specialization and limits the number of workers to whom a particular language would be of use; this requires a given worker to be familiar with many computer languages in order to handle the full scope of problems with which he is concerned.

Thus, high level procedural languages of a general nature such as FORTRAN IV and PL/1 must continue to be the primary tool of most computer users. However, these languages should be supplemented by others which are designed to handle a selected class of operations conveniently. One such language is MATLAN (Matrix Language), which was developed by IBM Germany for use with OS/360 and was released in December 1968.* The features of this language are such that it should prove to be extremely useful as a classroom and research tool in the study of matrix methods of structural analysis. The facility with which structural mechanics codes may be generated by students or researchers with limited programming experience will be demonstrated below.

MATRIX LANGUAGE

Basically, MATLAN is a programming system which is designed to simplify handling of and computation with matrices. The language contains provision for matrix

*System/360 Matrix Language (MATLAN) (360A-CM-05X), Program Description Manual.

generation, manipulation, arithmetic and program control, and may call MATLAN or FORTRAN subprograms. There are two primary features which make the language particularly attractive. The first is the extreme ease and conciseness with which matrix operations may be coded; this feature will be demonstrated in an example to follow. The second major advantage is that the automatic storage management scheme controls the dynamic flow of data within the system. It is responsible for dynamic allocation, renaming and canceling of matrices both in core and in a random access file. Matrices are automatically stored in array or coordinate form according to the density of data and are dynamically allocated to core or direct access file during execution. Two versions exist for all algorithms. One version, the core algorithm, occurs when sufficient core storage is available for the arrays; otherwise a segmenting algorithm is executed. Thus the matrix operations are not only simple to code, but are executed in a more efficient way than could be expected in a FORTRAN code written by one who, though competent in matrix concepts and in the application area, is a novice or occasional programmer.

STRUCTURAL ANALYSIS PROBLEM

In order to illustrate the application of MATLAN to structural analysis problems, a general method of elastic analysis of redundant structures by a matrix force method due to Wehle and Lansing (1952) has been coded. The resulting program is used to analyze an idealized wing structure.

Let the structure under consideration be divided into easily analyzed members, and let the loads acting on the individual members be given by $\{Q\} = \{Q'\} + \{Q''\}$, where $\{Q'\}$ represents s statically determinate loads and $\{Q''\}$ represents r redundant loads. Then s independent linear algebraic equations of equilibrium may be written for the structure. These equations may be solved for $\{Q'\}$ and combined with r trivial relations for the redundant loads to give the $s + r$ equations

$$\{Q\} = [G^1; G^2] \left\{ \begin{matrix} P \\ Q'' \end{matrix} \right\}, \quad (1)$$

where the P are known applied loads.

The redundant loads are determined by minimizing the strain energy

$$U = \frac{1}{2} [Q] [A] \{Q\} = \frac{1}{2} [P; Q''] [B] [P; Q'']^T \quad (2)$$

where $[A]$ is the influence coefficient matrix and

$$[B] = \begin{bmatrix} B^1 & B^2 \\ B^3 & B^4 \end{bmatrix} = [G^1; G^2]^T [A] [G^1; G^2]. \quad (3)$$

The energy is extremized by requiring the vanishing of $\partial U / \partial Q''$. The resulting equation in terms of $\{P\}$ and $\{Q''\}$ may be solved for $\{Q''\}$ to obtain

$$\{Q''\} = -[B^4]^{-1} [B^3] \{P\} = [C] \{P\}. \quad (4)$$

From (1) and (4)

$$\{Q\} = [G^1 + G^2 C] \{P\}. \quad (5)$$

To complete the solution, Castigliano's theorem is applied to obtain the displacements of each of the n applied loads P . Upon differentiating U with respect to the applied loads and making use of the solution for Q'' one finds

$$\{D\} = [B^1; B^2] \left[\frac{I}{C} \right] \{P\}. \quad (6)$$

Thus, given $[A]$ and $[G]$, the internal load distribution and deflections of the applied loads may be computed for any structure.

MATLAN CODE

The following code will implement the analysis described above:

1. MAIN	WEHLE
2. READ	(A,G)
3. HEADING	'STATIC INDETERMINATE STRUCTURE MATRIX SOLUTION'
4. TEXT	'SUM OF MEMBER FLEXIBILITIES', C
5. WRITE	A
6. NEWPAGE	
7. TEXT	'UNIT LOAD DISTRIBUTION', C
8. WRITE	G
9. NEWPAGE	
10. MULT	A,G,X
11. TRANS	G,GT
12. MULT	GT,X,B
13. EXSUBM	B,(1,1),(4,4),B1
14. EXSUBM	B,(1,5),(4,3),B2
15. EXSUBM	B,(5,1),(3,4),B3
16. EXSUBM	B,(5,5),(3,3),B4
17. DIV	B4,B3,Y
18. COPYC	Y,C
19. EXSUBM	G,(1,1),(17,4),G1


```

20. EXSUBM      G,(1,5),(17,3),G2
21. MULT       G2,C,Z
22. ADD        G1,Z,LOADS
23. MULT       B2,C,Z
24. ADD        B1,Z,INFCOF
25. TEXT       'REDUNDANT STRUCTURE UNIT LOAD
               DISTRIBUTION'
26. WRITE      LOADS
27. NEWPAGE
28. TEXT       'INFLUENCE COEFFICIENTS'
29. WRITE      INFCOF
30. END

```

In this program, the actual computations are described in statements 10-24, the remaining statements being devoted to input/output bookkeeping. Statements 10-12 compute the product indicated in Equation (3). Statements 13-16 and 19-20 partition [B] and [G] as indicated in Equation (3). The numerical parameters which appear in these statements correspond to the matrix sizes of the sample problem to be described below. In a working program, they would, of course, be treated as variables to be read in as data at execution time. Statements 17 and 18 perform the matrix inversion, multiplication and sign change indicated in Equation (4); 21 and 22 compute the load distribution according to Equation (5); and 23 and 24 compute the influence coefficients according to Equation (6). The advantages to the student of coding this analysis in MATLAN as opposed to FORTRAN or even PL/I are manifestly evident.

SAMPLE PROBLEM AND CONCLUSIONS

In order to test the program, the same example as employed in the original Wehle and Lansing (1952) paper was selected. A tapered swept-back wing was idealized by a conventional sheet-stringer box. This required the solution for 17 unknown loads in a thrice redundant structure with 4 applied loads. Thus $s = 14$, $r = 3$, $n = 4$ and [G] is a 17×17 matrix and [A] is a 17×17 matrix. The [A] matrix is sparsely populated so the MATLAN algorithms will automatically store it in compressed mode. Also [B⁴] can be shown to be symmetric and positive definite. These properties could have been specified in an ATTRIBUTE statement in the program, and the solution algorithm employed in the DIV instruction in statement 17 would have automatically employed an algorithm appropriate to such a matrix. Since such refinements were not incorporated into this sample program the solution proceeded as if the equations were asymmetric. The computations were carried out on an

IBM 360/65 computer running under MFT-II (Multi-programming with Fixed Tasks). The CPU time expended on the sample problem was approximately 7 seconds, including compile, link edit and execution.

This example illustrates the ease with which a structural analysis program can be written in MATLAN. The student never loses sight of the operation which is to be carried out by becoming involved in the details of that operation, e.g., the inversion of a matrix. With the availability of this new language, students without prior programming experience learn to apply matrix structural and analytical techniques to simple but realistic problems very quickly. In addition, the student will have been introduced to a useful engineering tool since the language allows one to handle extremely large matrices.

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THE APPLICATION OF ACCELERATION ANALYSIS TO THE COMPUTATION OF THE PRIMARY FLOW IN TURBOMACHINERY

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The absolute acceleration of a particle, $\bar{a}_A$, in motion with respect to a rotating reference frame consists of two components: the acceleration of the particle relative to its coincident point in the rotating reference frame and the absolute acceleration of the coincident point. Each of these two components is in turn equal to two physically identifiable components. The first component is equal to the acceleration of the particle relative to the rotating reference frame as if the frame were static, $\bar{a}_*$, and the Coriolis acceleration, $\bar{a}_{COR}$. The second component is equal to the centripetal acceleration, $\bar{a}_{CP}$, and the tangential acceleration, $\bar{a}_T$. Hence, the following equation is obtained:

$$\bar{a}_A = \bar{a}_* + \bar{a}_{COR} + \bar{a}_{CP} + \bar{a}_T. \quad [1]$$

This equation is commonly used in mechanics of particles and in mechanics of machines. It may also be applied to the fluid mechanics of turbomachinery enabling a formulation of the governing equations which has the following advantages over conventional formulations:

1. Greater insight into the mechanics of the flow is permitted since all terms are physically identifiable.
2. Comprehension of the concepts conveyed by equation [1] itself is aided.
3. Solution of the equations by numerical methods may be performed easily.

Although the writing of a complete program requires too much time for incorporation into a fluid mechanics course, a student may write one or more of the required sub-routines. The algorithm for the solution of the equations governing the flow is basically as follows:

1. A first approximation to the relative streamlines is calculated.
2. Flow angles, streamline curvatures and other necessary geometric parameters are calculated.

3. The relative total temperature and the relative total pressure distribution along streamlines are calculated by using the equation of conservation of energy, the Euler turbine equation, and the equation relating temperatures and pressures adiabatically.
4. The continuity equation and the Euler equation of motion are solved simultaneously along calculation lines, which cross the streamlines, to yield a new closer approximation to the streamlines and required aerodynamic parameters. The format of the Euler equation of motion used embodies equation [1] and is the factor which makes this analysis unique.
5. Using the new approximation to the streamlines, the foregoing procedure is repeated until the repeatability of results is within an acceptable tolerance.

In order to illustrate this application, the following is a formulation for two-dimensional radial turbomachines. (Similar formulations may be used for axial and mixed flow [three-dimensional] turbomachinery.)

A rotor is considered to consist of a series of channels formed by its blades. The flow in a single channel is to be analyzed. (Overall turbomachine parameters may be calculated from data obtained from the single channel analysis.) The channel is defined by the pressure surface of one blade and the suction surface of the next blade. In order to facilitate relating geometric and aerodynamic data and equations, three coordinate systems shown in Fig. 1 are required: a Cartesian and a cylindrical coordinate system having the center of rotation as the common origin, and a natural coordinate system. Calculation points, defined by the intersection of the relative streamlines (indexed i), and calculation lines of constant radius (indexed j), are established within the channel. The blade surfaces, i.e., the boundary streamlines, are specified numerically by arrays of Cartesian coordinates. The coordinates of the consecutive points on each blade surface are taken such that they lie on the same calculation line.

In order to maintain the same equations for both compressor and turbine geometries, the x component of the velocity is always taken to be in the positive x direction, therefore channels representing compressors and turbines are to be specified by positive and negative x coordinates respectively. Furthermore, the cylindrical coordinate systems must be different for compressor and turbine geometries, as indicated by the unit vectors $\bar{u}_r$ and $\bar{u}_\theta$.

shown in Fig. 1. Hence the radii of the calculation lines are assigned the sign of the x coordinate, i.e., r is taken positive for compressors and negative for turbines. All values of angles are taken between $-\pi/2$ and $+\pi/2$.

The coordinates of the first streamline approximation may be obtained by a linear interpolation of the calculation lines in cylindrical coordinates and then converting to Cartesian coordinates. Geometric parameters to be required in the analysis are calculated from the streamline coordinates. The relative flow angle, δ , the distance along the streamlines, s, and the relative streamline curvature, K, are calculated at each calculation point. The quantity $(dA/ds)/A$, where A is the area between and normal to the streamlines, is calculated for all calculation points except those on the blade surfaces. Required derivatives may be obtained using standard computer software for curvefitting and differentiation or other suitable methods. It has been found that good results may be obtained using a quadratic piecewise curvefitting algorithm to calculate the relative flow angle and thence the relative streamline curvature by $d\delta/ds$.

The pressure, temperature and other fields of the rotor are functions of time relative to an absolute reference frame. Since they are not time-dependent, but are steady state relative to the rotor, the analysis is performed relative to the rotor with the exception of the energy and momentum analysis which must be related to the absolute frame, since the rotating reference frame is non-Newtonian. The equations are therefore considered applicable instantaneously, that is, variations with respect to time in the absolute reference frame are not considered. It should be noted that the static pressure and static temperature are properties of the fluid and are thus the same in the rotating and absolute reference frames. Whereas the greater part of the analysis is performed relative to the rotor, quantities which may be either absolute or relative, such as pressure, temperature, velocity and Mach number, will only be subscripted A when absolute. Total conditions will be denoted by capitals and static conditions by lower case letters.

The following aerodynamic quantities are taken as specified: inlet relative total temperature, T_1 ; inlet relative total pressure, P_1 ; mass flowrate, $\dot{m}$; angular velocity of the rotor, ω ; angular acceleration of the rotor, $\dot{\omega}$; specific gas constant, R; ratio of specific heats, γ ; and specific heat at constant pressure, c_p .

The distribution of relative total pressure and relative total temperature is determined by relating downstream values to values at the initial or previous calculation lines. This is done by applying the Euler turbine equations as follows. The equation of the conservation of energy between two points along a given absolute streamline is

$$\left(c_p t + \frac{V_A^2}{2}\right)_j + E = \left(c_p t + \frac{V_A^2}{2}\right)_{j+1}, \quad [2]$$

where E is the energy transferred into (positive) the fluid per unit mass. The Euler turbine equation, which is a form of the equation of conservation of impulse and momentum, is

$$E = (UV_A \sin \alpha)_{j+1} - (UV_A \sin \alpha)_j. \quad [3]$$

Noting that $V_A \sin \alpha = U + V \sin \beta$, and

$$V_A^2 = V^2 + U^2 + 2UV \sin \beta, \quad [5]$$

where α and β are the absolute and relative flow angles with respect to the radial direction, respectively. It can be shown by combining the energy and the Euler turbine equation and eliminating V_A^2 and $V_A \sin \alpha$ that

$$\left(c_p t + \frac{V^2}{2} - \frac{U^2}{2}\right)_j = \left(c_p t + \frac{V^2}{2} - \frac{U^2}{2}\right)_{j+1} \quad [6]$$

which shows that the quantity

$$c_p t + \frac{V^2}{2} - \frac{U^2}{2}$$

is constant along a given streamline. Further, noting that

$$T = t + \frac{V^2}{2c_p} \quad \text{and} \quad U = \omega r \quad [7], [8]$$

and that the initial relative total temperature is specified, the relative total temperature at any radius may be determined by

$$T_{j+1} = \left(T_1 - \frac{\omega^2 r_1^2}{2c_p}\right) + \frac{\omega^2 r_{j+1}^2}{2c_p}. \quad [9]$$

Whereas the flow is assumed to be ideal, adiabatic, and the relative total temperature is a function of a radius only, the relative total pressure is also a function of radius only and it may be determined by

$$P_j = P_{j-1} \left(\frac{T_j}{T_{j-1}} \right)^{\frac{\gamma}{\gamma-1}}. \quad [10]$$

The static pressure and static temperature distributions along calculation lines are obtainable by means of radial equilibrium analysis. This consists of simultaneously solving, along a calculation line, the Euler equation of motion and the continuity equation having relative total pressure, relative total temperature, and geometric parameters as known quantities. The Euler equation of motion relating the static pressure and acceleration fields applied to a particle of fluid at a given instant is

$$-\frac{1}{\rho} \nabla p = \bar{a}_A. \quad [11]$$

The acceleration of particle of fluid under consideration, $\bar{a}_A$, may be formulated mathematically in several ways. This analysis is based on that formulation of $\bar{a}_A$ given by equation [1]. The magnitudes and direction of the acceleration components of $\bar{a}_A$ are as follows: The acceleration component $\bar{a}_A$ is further separable into two physically identifiable components, $\bar{a}_{*n}$ and $\bar{a}_{*s}$. $\bar{a}_{*n}$ is the component which is directed towards the center of curvature of the relative streamline and is normal to the relative velocity. The equation for $\bar{a}_{*n}$ is obtainable by applying Euler's equation of motion to the motion of a particle in a plane expressing the gradient operator in cylindrical natural coordinate form and is

$$\bar{a}_{*n} = \frac{V^2}{r_c} \bar{u}_n = KV^2 \bar{u}_n. \quad [12]$$

$\bar{a}_{*s}$ is in the direction of the relative velocity. The equation for $\bar{a}_{*s}$ is obtainable from the analysis of one-dimensional isentropic flow through a channel of varying flow area and is

$$\bar{a}_{*s} = V \frac{dV}{ds} \bar{u}_s = \frac{V^2}{M^2 - 1} \left(\frac{1}{A} \frac{dA}{ds} \right) \bar{u}_s. \quad [13]$$

The two foregoing equations are in terms of relative parameters, although the analysis for obtaining them is performed as if the reference frame were static. This is consistent with the concept of the acceleration component $\bar{a}_{*}$. Noting that $\bar{\omega} = \omega \bar{u}_z$, $\bar{V} = V \bar{u}_s$, $\bar{r} = r \bar{u}_r$, $\bar{\omega} = \dot{\omega} \bar{u}_z$ and the definitions of the unit vectors as per Fig. 1, the equations for the Coriolis, centripetal and tangential acceleration components are

$$\bar{a}_{COR} = 2\bar{\omega} \times \bar{V} = 2\omega V \bar{u}_n, \quad [14]$$

$$\bar{a}_{CP} = \bar{\omega} \times (\bar{\omega} \times \bar{r}) = -\omega^2 r \bar{u}_\theta, \quad \text{and} \quad [15]$$

$$\bar{a}_T = \dot{\bar{\omega}} \times \bar{r} = \dot{\omega} r \bar{u}_\theta, \quad [16]$$

respectively. The centripetal and tangential acceleration may each be separated into normal and streamwise components as follows:

$$\bar{a}_{CPn} = \omega^2 r \sin \beta \bar{u}_n \quad [17]$$

$$\bar{a}_{CPS} = -\omega^2 r \cos \beta \bar{u}_s \quad [18]$$

$$\bar{a}_{Tn} = \dot{\omega} r \cos \beta \bar{u}_n \quad [19]$$

$$\bar{a}_{Ts} = \dot{\omega} r \sin \beta \bar{u}_s \quad [20]$$

Adding acceleration components having like unit vectors yields

$$\text{for } \bar{u}_s, \quad \bar{a}_{*s} + \bar{a}_{CPS} + \bar{a}_{Ts} = \bar{a}_s, \quad \text{and} \quad [21]$$

$$\text{for } \bar{u}_n, \quad \bar{a}_{*n} + \bar{a}_{CPn} + \bar{a}_{COR} + \bar{a}_{Tn} = \bar{a}_n. \quad [22]$$

Substituting the foregoing into the Euler's equation of motion and equating terms of like unit vectors yields

$$\frac{\partial p}{\partial s} \bar{u}_s = -\rho \bar{a}_s, \quad \text{and} \quad [23]$$

$$\frac{\partial p}{\partial n} \bar{u}_n = -\rho \bar{a}_n. \quad [24]$$

The differential of static pressure along a calculation line between two consecutive streamlines is

$$dp = \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial n} dn. \quad [25]$$

The differentials ds and dn , in terms of dx and dy are

$$ds = dx \cos \delta + dy \sin \delta, \text{ and} \quad [26]$$

$$dn = -dx \sin \delta + dy \cos \delta. \quad [27]$$

The static pressure at an intersection point of a streamline and a given calculation line is obtainable in terms of the static pressure at the intersection point of the previous streamline and the same calculation line by substituting the appropriate terms of the foregoing equations in

$$p_{i+1,j} = p_{i,j} + dp. \quad [28]$$

In order to begin computing the static pressure distribution along a calculation line, the static pressure at the one calculation point on the same line must be known. Whereas static pressures are not specified, a static pressure or parameter enabling the calculation of a static pressure must be assumed. It is best to assume the relative Mach number at the initial calculation point on a calculation line and calculate the static pressure by

$$p = P \left(1 + \frac{\gamma-1}{2} M^2 \right)^{-\frac{\gamma}{\gamma-1}} \quad [29]$$

It has been found desirable to choose the central streamline, rather than the streamline indexed 1, as the initial streamline on which the Mach number is assumed and integrate outward toward the blade surfaces. The static temperature, relative velocity and density at that point may be calculated by

$$t = T / \left(1 + \frac{\gamma-1}{2} M^2 \right), \quad [30]$$

$$V = M \sqrt{\gamma R t}, \text{ and} \quad [31]$$

$$\rho = p/Rt. \quad [32]$$

The static pressure differential and the static pressure on the next point on the calculation line may now be calculated using equations [12] to [29] and

$$M = \left(\frac{2}{\gamma-1} \left(\left(\frac{p}{P} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right) \right)^{1/2}. \quad [33]$$

The relative Mach number at the next point is now known and the procedure may be repeated from point to point until the static pressure, static temperature and relative velocity distributions along the calculation line are obtained. Using the continuity equation

$$\dot{m} = \int \rho V dA, \quad [34]$$

the total mass flowrate is obtained by integration along the calculation line. If the calculated mass flowrate is not equal to the specified mass flowrate, $\dot{m}_{REQ}$, within an acceptable tolerance, a new estimate of the relative Mach number is assumed, noting that

$$\frac{\dot{m}}{A} = \frac{\gamma P^2}{RT} f(M), \quad [35]$$

$$\text{where} \quad f(M) = M \left(1 + \frac{\gamma-1}{2} M^2 \right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

$$\text{from} \quad f(M)_{NEW} = f(M)_{OLD} \frac{\dot{m}_{REQ}}{\dot{m}_{OLD}}$$

The values subscripted "OLD" are those based on the previous estimate of Mach number. This procedure is repeated until the calculated mass flowrate lies within an acceptable tolerance. This procedure yields the simultaneous solution of the continuity equation and the Euler equation. If the positions of the streamlines used are not a good approximation, the mass flowrate between adjacent streamlines will not be equal. New positions of the calculation points are computed such that the mass flowrate between streamlines is equal. The entire procedure is repeated successively until the repeatability of the parameters at each calculation point lies within an acceptable tolerance. The parameters selected for the repeatability check may be fundamental parameters, such as streamline positions, or required parameters, such as static pressure, static temperature and relative velocity. The root mean square of the relative velocity change between successive iterations is found to be most practical. This quantity may also be used for a convergence check between successive even iterations. Repeatability of parameters signifies an

approximation to the solution of the governing aerodynamic and geometric equations. The iterative process of determining the streamline positions may diverge if the previous approximation of the positions is a poor one or if the change between successive iterations is too large. The use of "damping factors," which limit the changes of the streamline positions, curvatures, and the quantities dA/ds , prove useful in attaining convergence. When using damping factors, a minimum number of iterations, 4 to 6, should be allowed before testing for repeatability to ascertain that repeatability is due to the attainment of a solution and not to excessive damping. It is further desirable to increase mass flowrate and angular velocity of the rotor in steps until desired values are attained.

The accuracy of a numerical solution may be assessed in several ways. For flows in static channels, a numerical solution may be compared with a readily available analytical solution, such as a solution for the flow in a channel bounded by logarithmic spirals. Whereas analytical solutions for flows in rotating channels are not readily available, the accuracy of a numerical solution may be assessed by comparing the power input or output calculated by three different methods: by energy, by momentum and by integrating the static pressure on the blade surfaces. Whereas the law of conservation of energy and the law of conservation of impulse and momentum are applied using data along inlet and outlet calculation lines and the static pressure integration procedure is applied using data along bounding streamlines, good agreement between the three values of the power tends to indicate a good solution, although this does not constitute an absolute proof. The calculation of the torque about the centerline of rotation by two different methods may also be used for assessing the accuracy of solutions for both static and rotating channels.

Whereas the analysis assumes adiabatic inviscous ideal flow--i.e., no losses, boundary layer, secondary flows--the solution is one for ideal primary flow only. Under most circumstances the primary flow will be perturbed by secondary effects; however, primary flow characteristics will predominate. As the magnitude of the perturbations increases, secondary effects may eventually predominate, e.g., flow separation. However, the primary flow does indicate basic characteristics of the flow in turbomachines and may serve as a basis for analysis of non-ideal flow. Probable boundary layer migration and flow separation may be deduced from the static pressure and relative velocities fields of the ideal flow. It should be noted that total pressure losses in real flow

will affect the nature of the flow. Methods of including total pressure losses in the analysis are currently being investigated by the author. It should also be noted that boundary conditions have been implicitly determined by the algorithm for determination of relative streamline curvatures. This algorithm yields good results as indicated by the power criteria outlined. A detailed analysis of boundary conditions is beyond the scope of this paper. Such an analysis must consider the flow outside the rotor. Attempts by the author at forcing compatibility between the flows inside and outside of the rotor by forcing uniform inlet, or both inlet and outlet, relative velocities have been found to yield poor results as indicated by the power criteria outlined. It is the opinion of the author that the interaction between the flows inside and outside of the rotor is one of the justifications for the assumption of circumferentially constant inlet relative total temperature and pressure. Circumferential variations in inlet relative total temperature and pressure may be based on circumferentially constant inlet absolute total temperature and pressure. Some references on experimental work which give some insight into real flows are included in the reference section.

Blade surface relative velocity distributions and relative streamlines calculated for a rotor having 45 degree logarithmic spiral blading are shown in Fig. 2. This and other such data obtained from a program based on the method of analysis outlined in this paper is useful in the discussion of the mechanics of turbomachinery from both the theoretical and practical point of view. The analysis, although based on equations usually studied in elementary courses on machines and on fluids, is similar to the type of analysis which might be performed by one in industry and may help bridge the gap between textbook and "real-world" problems.

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NOTE: A computer program for the above analysis using a CDC computer may be obtained by writing to the author.

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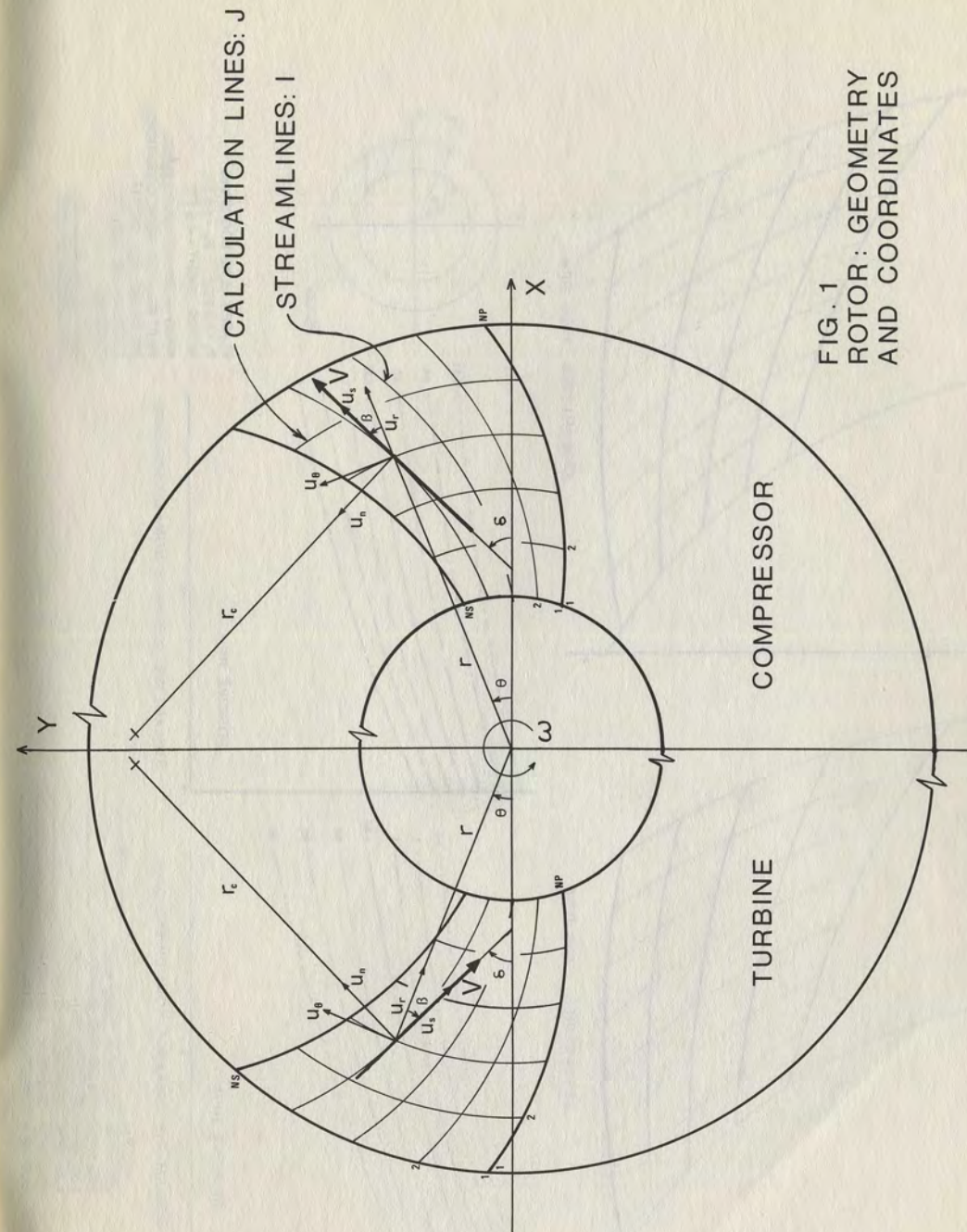
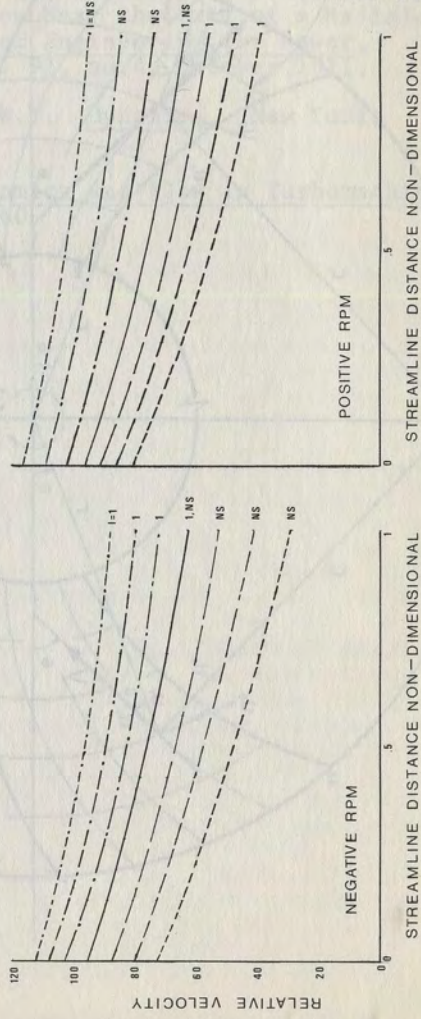
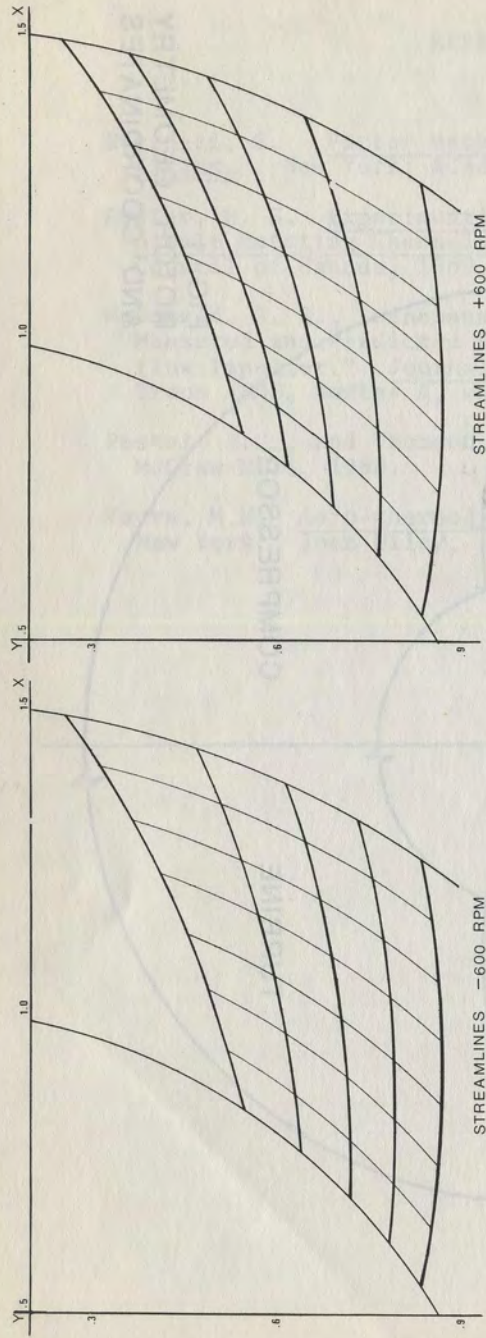
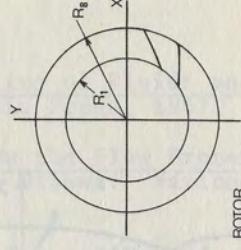


FIG. 1
ROTOR: GEOMETRY
AND COORDINATES



0 RPM SUCTION SURFACE
 200 RPM PRESSURE SURFACE
 400 RPM SUCTION SURFACE
 600 RPM PRESSURE SURFACE
 800 RPM SUCTION SURFACE
 800 RPM PRESSURE SURFACE



BLADES: 15, 45° LOG SPIRAL
 $R_1 = 1.0$ FT. $R_2 = 1.5$ FT.
 ROTOR HEIGHT = 0.5 FT.
 MASS FLOWRATE = 0.5 SLUG/SEC.
 INLET RELATIVE TOTAL CONDITIONS:
 PRESSURE: 2120 LB/FT²
 TEMPERATURE: 520 °R
 FLUID: AIR

FIG. 2 COMPUTED DATA